JET PROPULSION LABORATORY **INTEROFFICE MEMORANDUM**

3454–97–018

TO: Distribution
FROM: Mark Milman
SUBJECT: Analysis building blocks for the SIM Simulator

Introduction. This short memo presents an overview of my perspective of how SIM operates in the astrometry mode. This perspective is limited to gross geometrical and operational considerations necessary for astrometry data acquisition. The discussion is not intended to be complete, but hopefully will shed some light on issues related to the SIM SIMULATOR, and provide a framework (viz. a viz. the relevant equations) with which to evaluate them. Critical to this development is the notion of the "regularized" measurement. This is the primary output of the astrometry data acquisition process and the foundation of the astrometric equations. The role of the external metrology system and how it interacts with the instrument flexiblity in this process will also be developed. We will walk through various observation scenarios, beginning from a simplified set of assumptions to a more complete set to elucidate this concept. A few model and observability assumptions (some may be implicit) are made along the way to make everything hang together. The thoughts here are somewhat preliminary, but nevertheless it is worthwhile to present them at this early stage.

To begin it is important to define the data set and variables in an initially simplified and pristine form that define the astrometry equations from which some of the important science objectives are derived. Let s_0, s_1, ..., s_N denote the directions of $N+1$ stars, and let b_1,...,b_M denote M interferometer baseline vectors. Assume perfect astrometric measurements of the form

$$d_{ij} = < s_i, b_j > . \tag{1}$$

(It is not necessary that each star is observed by the same set of baselines above, i.e. there are not necessarily $M(N+1)$ measurements.) Because of *a priori* knowledge errors in the star directions and baseline vectors we look for corrections ω_i, ω^j of the form

$$s_i \to s_i + \omega_i \times s_i, \quad i = 0, ..., N$$

and

$$b_j \to b_j + \omega^j \times b_j \quad j = 1, ..., M.$$

The small rotations ω_i (ω^j) correspond to displacements in the tangent planes to the unit sphere at s_i (b_j). Solving for the $\{\omega_i\}$ is one of the major science objectives of global astrometry.

To first order, (1) becomes

$$y_{ij} = < s_i, b_j > + < s_i \times b_j, \omega_i > - < s_i \times b_j, \omega^j > . \tag{2}$$

The dimension of each of the tangent planes is 2, and thus each rotation defines two degrees of freedom. It then becomes evident that there must be multiple star measurements for each baseline, for otherwise the system of equations in (2) will be underdetermined. This is a fundamental caveat of the SIM observation process.

Although the solvability of the system of equations (2) is not pursued here, we note that it is not difficult to show that the null space of the matrix corresponding to this system contains (at least) three vectors corresponding to the infintesimal global rotations of the sphere. And with proper connectivity of the observation sequence over the celestial sphere these infinitesimal rotations can be shown to span the null space of this matrix.

There are other astrometric and instrument parameters of interest. The discussion will not address the astrometric parameters of parallax and proper motion, but some of the instrument parameters that arise as part of the astrometric reconstruction problem will be. For example, since the absolute lengths of the interferometer baselines are only known to an accuracy of several microns, these parameters emerge as instrument parameters that must be estimated within the context of the astrometric equations. The analysis presented herein also shows that the intial alignments of the interferometer baslelines must likewise be estimated. It is probably a safe bet that other parameters will have to be estimated as the analysis incorporates more models of the various subsystems.

Bright Science Stars with Rigid Spacecraft. The astrometry equations in (1) represent the "perfect" scenario in which *instantaneous* measurements are taken of multiple stars with a single baseline. Of course this is not possible, since even when the science star is bright enough to close the delay line control loops around the science star signal itself, the instrument baseline is not fixed in inertial space over the integration time. We first investigate how this affects the astrometric equation (1) under the assumption that the spacecraft is rigid and the science star is bright (i.e., no need for feedforward).

Let $\{E_j\}$ denote the inertial frame, and let $\{E_j(t)\}$ denote a spacecraft fixed frame. Let the guide interferometer baselines be denoted b_1 and b_2, the science interferometer baseline is represented as b_3, and the corresponding guide and science star directions are given as s_1, s_2, and s_3, respectively. To reflect the time–varying nature of the baselines we write (1) as

$$y_i(t) = < s_i, b_i(t) >, \tag{3}$$

where in accordance with the rigid spacecraft assumption,

$$b_i(t) = \sum b_i^j E_j(t), \tag{4}$$

with the components b_i^j *constant* with respect to the time varying frame $E_j(t)$. We will always assume that the time–varying frame $E_j(t)$ is close to the inertial frame (so that the approximation $E_j(t) = E_j + \omega \times E_j$ for some vector ω is valid to a several picoradian accuracy). Note that this relative alignment of the two frames is not a knowledge assumption, but actually a *stability* assumption. Incorporating this time–varying component of the problem, (2) becomes

$$y_i(t) = < \hat{s}_i, b_i > + < \hat{s}_i \times b_i, \omega_i > + < \hat{s}_i \times b_i, \omega(t) > + < s_i \times b_i, \omega_i \times \omega(t) > \qquad (5).$$

The second order term above becomes negligible when the the product of the stability error and guide star position error satisfy an approximate bound

$$|\omega(t)||\omega_i||b_i| \approx 10^{-11}\text{m}. \tag{6}$$

(I haven't meticulously gone through the error budget here, but the bound in (6) looks achievable and in the right ballpark.) Even after discarding the second order term, the measurement equation is still time dependent because of the presence of $\omega(t)$. But if the delay measurement is simply averaged, we obtain an equation similar to (2) of the form

$$\bar{y}_i = < \hat{s}_i, b_i > + < \hat{s}_i \times b_i, \omega_i > + < \hat{s}_i \times b_i, \bar{\omega} >, \tag{7}$$

where

$$\bar{y}_i = \frac{1}{N} \sum_k^N y_i(t_k), \quad \text{and} \quad \bar{\omega} = \frac{1}{N} \sum_k^N \omega(t_k). \tag{8}$$

Equation (7) is of the same character as (2), with the exceptions that the instantaneous angular rotation has been replaced by the time–averaged rotation and the delay measurement has been averaged. Thus the regularized measurement in this scenario is simply defined as the average of the internal metrology delay measurements.

Under the present assumptions, the observation caveat is fulfilled as three star observations are obtained for each baseline since only the estimate of the average rotation $\bar{\omega}$ common to the two guide star and science star interferometers is required to determine the baseline orientations in inertial space. This is essentially the scenario considered in the global astrometric study in [2]. These conditions are overly restrictive, requiring observations of only bright stars.

Dim Star with Rigid Spacecraft. When the science star is too dim to close the delay line controller around the fringe signal, feedforward is used, and a different form of the regularized measurement is needed.

The delay measurement equation (5) defines exactly what the feedforward command should look like. Let $z(t)$ denote this command. Then z is generated from the known and approximate quantities in (5) via

$$z(t) =< \hat{s}_3, b_3 > + < b_3 \times \hat{s}_3, \hat{\omega}(t) >, \tag{9}$$

where $\hat{\omega}$ denotes the estimate of ω. Contrast this with the "exact" feedforward term (excluding second order terms)

$$y_3(t) =< \hat{s}_3, b_3 > + < b_3 \times \hat{s}_3, \omega_3 > + < b_3 \times \hat{s}_3, \omega(t) > . \tag{10}$$

The difference, d_3, of these two terms is actually measured with the science star fringe:

$$d_3(t) =< \hat{s}_3 \times b_3, \omega_3 > + < b_3 \times \hat{s}_3, \omega(t) - \hat{\omega}(t) > . \tag{11}$$

At first inspection the apparent time–varying nature of (11) doesn't bode well for producing fringes. But as we shall see $\omega(t) - \hat{\omega}(t)$ (excluding second order terms), is actually time–invariant. And not only is $\omega(t) - \hat{\omega}(t)$ constant, its value depends only on the guide star interferometer baselines and guide stars. This is a critical point since if the science interferometer siderostats point at another star, this term still does not change, and SIM can effectively achieve the required multiple star measurements with a single baseline orientation. We fill in the gaps of this discussion with a little anlaysis below.

The estimate $\hat{\omega}(t)$ is synthesized as

$$\hat{\omega} = T^{\dagger}(y - Sb), \tag{12}$$

where the matrix T is defined as

$$T = \begin{pmatrix} (\hat{s}_1 \times b_1)^T \\ (\hat{s}_1 \times b_1)^T \end{pmatrix}, \tag{13}$$

$T^{\dagger}$ denotes the pseudoinverse of T,

$$Sb = \begin{pmatrix} < \hat{s}_1, b_1 > \\ < \hat{s}_2, b_2 > \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{14}$$

Expanding (5), we see to first order that ω satisfies

$$\omega = T^{\dagger}(y - Sb - v) + \Pi\omega, \tag{14}$$

where Π denotes the orthogonal projection onto the null space of T and

$$v = \begin{pmatrix} < s_1 \times b_1, \omega_1 > \\ < s_2 \times b_2, \omega_2 > \end{pmatrix}. \tag{15}$$

Observe then that

$$\omega - \hat{\omega} = T^{\dagger} v + \Pi\omega. \tag{16}$$

Now $T^{\dagger} v$ is time invariant and depends only on the fixed vectors $\hat{s}_i$, b_i. The analysis in [1] shows that the time–varying component $\Pi\omega$ in (16) is a second order term so long as the interferometer baselines are sufficiently colinear and the elevation angle separation between the two guide stars is comparable to the largest of the separation angles between each guide star and science star (a mild constraint that should be easily met).

To first order, the delay equation from (11) takes the form

$$d_3 =< \hat{s}_3 \times b_3, \omega_3 > + < b_3 \times \hat{s}_3, \Delta\omega >, \tag{17}$$

where $\Delta\omega$ is the time invariant component of the difference between the estimated and actual rotation between the body fixed and inertial frames. Thus in the dim science star observation scenario, the regularized measurement is the feedforward stabilized fringe measurement, and instead of estimating the rotation vector between frames, the difference between the actual and estimated rotation used in the feedforward command is estimated in the astrometric equation.

Scale and Flexibility Errors. There are other error sources that must be accounted for in the astrometry equations for both the bright and dim star scenarios. The first is that the absolute position of the interferometer baselines with respect to the spacecraft frame are known only through absolute metrology to several microns. The error in the length of the baseline is refered to as the scale error. There is also an alignment error. Another error source derives from the flexibility of the spacecraft so that the components of the vectors in (4) are not actually constant. A linear flexible model of the baseline vectors would realize these vectors as a superposition of the flexible and rigid body modes of the system. In the inertial frame the baseline vector may be decomposed as

$$b_i(t) = b_i + \omega(t) \times b_i + \rho_i(t), \tag{18}$$

where the rotational component of the rigid body modes is absorbed in the expression $\omega(t) \times b_i$, and the flexible and translational modes are in $\rho_i(t)$. Note that $\omega(t)$ is the same for all of the interferometer baselines because it is the rigid body behavior of the spacecraft. The way SIM handles the flexibility component is to measure it – the external metrology measurements are used to estimate ρ_i. This process is quite important, so a short digression will be made to discuss how this can be done, and what additional error sources arise from this. Essentially what happens is that the external metrology yields a model of the form (18), with the important feature being that the rotation vector ω is the same for all baselines. However, as we shall see, this rotation vector is not precisely the one in (18), and the resulting model does not arise from any mechanical considerations.

Let $X_1, .., X_N$ denote corner cube locations. Let b_{ij} denote the baseline vector

$$b_{ij} = X_i - X_j \tag{19}$$

and write

$$X_i = X_i^0 + \Delta X_i,$$

$$X_j = X_j^0 + \Delta X_j,$$

where X_i^0 and X_j^0 are the nominal positions of the corner cubes, and $\Delta X_i, \Delta X_j$ are the perturbations from nominal. Thus

$$b_{ij} = b_{ij}^0 + \Delta X_i - \Delta X_j, \quad \text{where} \quad b_{ij}^0 = X_i^0 - X_j^0 \tag{20}$$

The external metrology system makes differential measurements of the form

$$\delta d_{ij} = X_k - X_i| - |X_k^0 - X_i^0| \quad k = 1, ..., 4; \quad i \neq k \tag{21}$$

where the $X_k, \quad k = 1, ..., 4$ are locations on the metrology tetrahedron. Following [3], let $F(x)$ denote the distance function between all measured vector pairs, i.e. each component function of F has the form $|X_k - X_j|$. Then to first order

$$|X_k - X_i| - |X_k^0 - X_i^0| = \frac{\partial F_{ki}}{\partial X_k}(\Delta X_k) + \frac{\partial F_{ki}}{\partial X_j}(\Delta X_i), \tag{22}$$

where

$$\frac{\partial F_{ki}}{\partial X_k} = \frac{1}{|X_k - X_i|} < X_k - X_i, \cdot >,$$

and

$$\frac{\partial F_{ki}}{\partial X_i} = \frac{1}{|X_k - X_i|} < X_i - X_k, \cdot > .$$

Hence the differential measurement is to first order

$$|X_k - X_i| - |X_k^0 - X_i^0| = \frac{1}{|X_k - X_i|} < X_k - X_i, \Delta X_k - \Delta X_i > . \tag{23}$$

In [3] this set of equations is written as

$$D_x F \Delta x = \delta d \quad \Delta x = \begin{pmatrix} \Delta X_1 \\ \vdots \\ \Delta X_N \end{pmatrix} \tag{24}$$

where $D_x F$ is the differential of F and δd is the vector consisting of the differential measurements. The null space of $D_x F$ consists of the rigid body translations and rotations [3]. Let δx denote the estimate of Δx obtained via the pseudoinverse solution

$$\delta x = D_x F^{\dagger} \delta d. \tag{25}$$

Since δx has no component in the null space of $D_x F$, there exist vectors h and ω ("translation" and "rotation" vectors, respectively) such that

$$\delta x - \Delta x = \begin{pmatrix} h + \omega \times X_1 \\ \vdots \\ h + \omega \times X_N \end{pmatrix} . \tag{26}$$

Let $\hat{b}_{ij} = b_{ij}^0 + \delta x_i - \delta x_j$ (using the obvious correspondence for the δx_i's), and note that

$$b_{ij} - \hat{b}_{ij} = \omega \times b_{ij}^0. \tag{27}$$

Hence,

$$b_{ij} = b_{ij}^0 + \omega \times b_{ij}^0 + \rho_i; \quad \rho_i = \delta x_i - \delta x_j. \tag{28}$$

Note that ω is independent of the baseline and the δx_i's are obtained from the external metrology data. This is the model analogous to (18) that meets the astrometry requirements. For the interested reader's information, the difference between these two models is that in reality the flexible component ρ is not orthogonal to the rigid body components, whereas in (28) there is orthogonality. (Orthogonality holds in (18) over all mechanical degrees of freedom, and then only with respect to the system mass matrix.)

Two errors arise from this solution. The first has to do with the linearization process, which will not be addressed here. The second error source results from using the inexact differential $D_x F$ in the computation (25). This is due to the absolute metrology error in the location of the nominal corner cube locations. Let X^0 denote the nominal estimate of the vector of corner cube positions, and let $\bar{X}$ denote their true positions. Given the differential measurement δd, the error in the computation of δx, call it ϵ, resulting from the absolute metrology error is

$$\begin{aligned} |\epsilon| &= |(D_{X^0} F^\dagger - D_{\bar{X}} F^\dagger)\delta d| \\ &= |(D_{X^0} F^\dagger - D_{\bar{X}} F^\dagger)||\delta d|. \end{aligned} \tag{29}$$

Further bounds can be obtained using

$$|(D_{X^0} F^\dagger - D_{\bar{X}} F^\dagger)| \leq 2|D_{X^0} F - D_{\bar{X}} F|_F |D_{X^0} F^\dagger|.$$

For the SIM configuration $|D_{X^0} F^\dagger| \approx 7$ and

$$|D_{X^0} F - D_{\bar{X}} F|_F \approx 6 * |b_{ij} - b_{ij}^0| / |b_{ij}^0|.$$

These bounds are not sharp, however they are probably representative of worst case errors. For example, a 10μm absolute metrology error coupled with a 10μm change in the length of the baselines could produce an error in the baseline measurement of approximately .75 nanometer. (We shouldn't sweat these numbers quite yet since they are preliminary.)

Next because of the initial error in the estimate of b_i, we write in the spacecraft fixed frame

$$b_i^0 = \hat{b}_i + \epsilon_i \hat{b}_i + \omega^i \times \hat{b}_i. \tag{30}$$

The error correction terms ϵ_i and ω^i appear because of the initial uncertainty in the external metrology measurements (which is on the order of several μm). It is assumed that ϵ_i and ω^i are fixed, but unknown constants. Using (30) in (28) gives

$$b_i(t) = \hat{b}_i + \epsilon_i \hat{b}_i + (\omega^i + \omega(t)) \times \hat{b}_i + \rho_i(t). \tag{31}$$

And retaining first order terms yields the astrometry equation

$$y_i(t) = < \hat{s}_i, \hat{b}_i > + < \hat{s}_i, \rho_i(t) > + \epsilon_i < \hat{s}_i, \hat{b}_i > + < \hat{s}_i \times \hat{b}_i, \omega_i > + < \hat{b}_i \times \hat{s}_i, (\omega^i + \omega(t) > . \tag{32}$$

First we will assume the stars are bright, making this the analogue of (7). Introducing the time averaged quantities

$$\bar{y}_i = \frac{1}{N} \sum_j y_i(t_j), \quad \bar{\omega} = \frac{1}{N} \sum_j \omega(t_j), \quad \text{and} \quad \bar{\rho}_i = \frac{1}{N} \sum_j \rho_i(t_j), \tag{33}$$

results in the astrometric equation

$$\bar{y}_i =< \hat{s}_i, \hat{b}_i > + < \hat{s}_i, \bar{\rho}_i > +\epsilon_i < \hat{s}_i, \hat{b}_i > + < \hat{s}_i \times \hat{b}_i, \omega_i > + < \hat{b}_i \times \hat{s}_i, (\omega^i + \bar{\omega}) > \tag{34}$$

with the regularized measurements $\bar{y}_i$. The assumption we make is that $\rho_i(t)$ is measured with the external metrology so that the term $< \hat{s}_i, \bar{\rho}_i >$ in (34) above is known. The other differences between this equation and (7) are the presence of the unknown scale term and misalignment terms, ϵ and ω^i. These errors are assumed fixed with respect to the spacecraft frame. With this assumption ϵ_i and ω^i emerge as instrument parameters that must be estimated, while $\bar{\omega}$ is the ideal time averaged orientation common for each baseline that must be estimated just as before. Note that by the way the misalignments enter into the astrometric equations (for a fixed star we can essentially only estimate the sum of the misalignment and rotation vectors), it would take observations of different stars to estimate these parameters.

Now let's see how misalignments and flexibility impact the feedforward scenario for dim science stars. We follow the outline provided by (9)–(17), while incorporating the additional terms introduced in the observation equation (32) due to baseline misalignments, scale error and flexibility. This time the synthesized feedforward command has the form

$$z =< \hat{s}_3, \hat{b}_3 > + < \hat{s}_3 \times \hat{b}_3, \hat{\omega}(t) > + < \hat{s}_3, \rho_3(t) >, \tag{35}$$

where $\hat{\omega}(t)$ is again the estimated rotation between the spacecraft and inertial frames. Using (32), the difference between the exact linearized feedforward term, y_3, and the estimated term z in (35) is

$$y_3 - z = \epsilon_3 < \hat{s}_3, \hat{b}_3 > + < \hat{s}_3 \times \hat{b}_3, \omega^3 > + < \hat{s}_3 \times \hat{b}_3, \omega(t) - \hat{\omega}(t) > + < \hat{b}_3 \times \hat{s}_3, \omega_3 > . \tag{36}$$

Note that $y_3 - z$ is measured with the science star fringe. Computing as before, we obtain

$$\hat{\omega}(t) = T^{\dagger}[y - Sb - S\rho] \tag{37}$$

keeping in mind that $\rho(t)$ is estimated by the external metrology. Including the additional terms in (32) to determine ω exactly (modulo its component in $N(T)$), the difference between the exact and estimated rotations is

$$\omega(t) - \hat{\omega}(t) = -T^{\dagger}[\epsilon Sb + v + w] + \Pi\omega(t), \tag{38}$$

where v is defined as in (15), Π is the orthogonal projection in (16), $\epsilon = diag(\epsilon_1, \epsilon_2)$, and

$$w = \begin{pmatrix} < s_1 \times b_1, \omega^1 > \\ < s_2 \times b_2, \omega^2 > \end{pmatrix} . \tag{39}$$

One of the keys to the stability of the science star fringe in the feedforward case with flexibility is the ability of the external metrology system to accurately measure $\rho(t)$. From (36) we obtain the astrometric equation (keeping in mind that $\omega(t) - \hat{\omega}(t)$ is time invariant)

$$y_3 - z = \epsilon_3 < \hat{s}_3, \hat{b}_3 > + < \hat{s}_3 \times \hat{b}_3, \omega^3 > + < \hat{s}_3 \times \hat{b}_3, \Delta\omega > + < \hat{b}_3 \times \hat{s}_3, \omega_3 > . \tag{40}$$

This equation is identical to (17) with the exceptions that it is now necessary to estimate the science interferometer baseline length and misalignment parameters, ϵ_3 and ω^3, respectively.

What's Next? Before the SIM SIMULATOR investigates the finer instrument errors (e.g., those characteristics that produce metrology errors, noncoincidence of starlight and metrology paths, etc.) it is advisable to firmly establish that the analysis we have presented is a realizable functional representation of how SIM operates at the coarse level. This means to make sure that we have identified appropriate instrument parameters, that they are indeed identifiable, and the notions of regularized measurements as defined make sense (i.e., they are in fact realizable with the instrument). We have also been somewhat glib in our use of time–averaging signals. A better model/quantification of these is also needed.

A couple of specific things: Assemble a SIM "geometry" model to verify and quantify the analysis presented here. Get a better handle on some of the time–varying aspects of the problem (e.g., how does motion actually affect the regularized measurements? What does the time–averaging really accomplish?) Since the real hand off between the instrument and the ground processing is the regularized measurement, a model of how the fringe position is processed would be useful. Also, as a very near term exercise, a revisit of the absolute metrology error is recommended.

References

[1] M. Milman, IOM 3456–96–029.

[2] S. Loiseau and F. Malbet, Global astrometry with OSI, Astron. Astrophys. Suppl. Ser. 116, pp. 373–380, (1996).

[3] M. Milman, IOM 345–97–015.

Distribution (Usual Suspects)

S. Bassinger
A. Boden
M. Colavita
S. Joshi
R. Laskin
K. Lau
M. Levine
J. Melody
L. Needels
D. Redding
S. Shaklan
S. Umwin
J. Yu